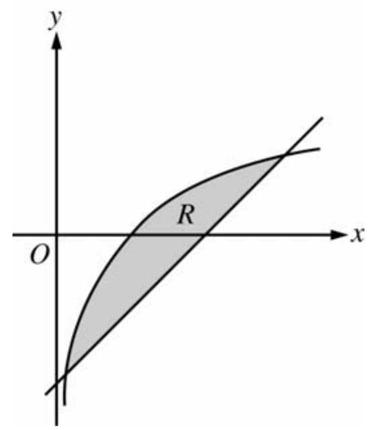
Question 1

Let R be the shaded region bounded by the graph of $y = \ln x$ and the line $y = x - 2$, as shown above.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the horizontal line $y = -3$.

(c) Write, but do not evaluate, an integral expression that can be used to find the volume of the solid generated when R is rotated about the y-axis.



$\ln(x) = x - 2$ when $x = 0.15859$ and 3.14619.
Let $S = 0.15859$ and $T = 3.14619$

(a) Area of $R = \int_S^T (\ln(x) - (x-2))\,dx = 1.949$	$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$
(b) Volume $= \pi \int_S^T \left((\ln(x) + 3)^2 - (x - 2 + 3)^2 \right) dx$ $= 34.198$ or 34.199	$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits, constant, and answer} \end{cases}$
(c) Volume $= \pi \int_{S-2}^{T-2} \left((y+2)^2 - \left(e^y\right)^2 \right) dy$	$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$

1A1

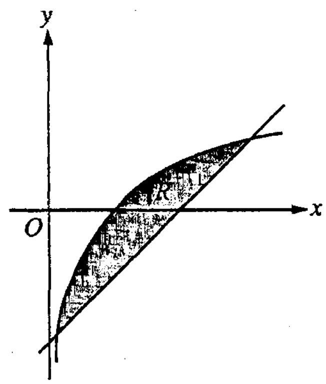
CALCULUS AB

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.



Work for problem 1(a)

$y = \ln x \qquad y = x - 2$

$\ln x = x - 2 \qquad x = .1586,\ x = 3.1462$

$$\int_{.1586}^{3.1462} (\ln x - x + 2)\, dx = \boxed{1.9491}$$

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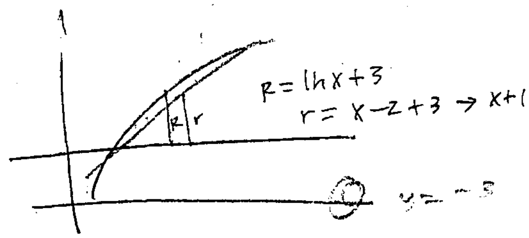
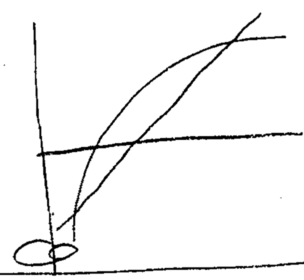
Continue problem 1 on page 5.

Work for problem 1(b)



$$\pi \int_{.1586}^{3.1462} \left[(\ln x+3)^2-(x-2+3)^2\right] dx = \boxed{34.1986}$$

Work for problem 1(c)



$$\boxed{2\pi \int_{.1586}^{3.1462} \left[x(\ln x-x+2)\right] dx}$$

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1B,

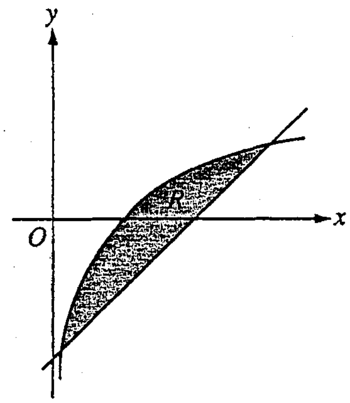
CALCULUS BC

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.



Work for problem 1(a)

$y = \ln x \quad y = x - 2$

$\ln x = x - 2 \quad x = .158594 + 3.14619$

$A = \int_{.158594}^{3.14619} (\ln x - (x-2))\,dx$

$A = 1.949$

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Continue problem 1 on page 5.

1B2

Work for problem 1(b)

$V = \pi \int_{.158594}^{3.14619} ((-3-\ln x)^2 - (-3-(x-2))^2)\,dx$

$V = 10.886\pi$

$V = 34.199$

Work for problem 1(c)

$x-2=0$ $\quad$ $\ln x = 0$

$x=2$ $\quad$ $x=1$

$V = \pi \int_{1}^{3.14619} ((\ln x)^2 - (x-2)^2)$

$V = \pi \int_{.158594}^{2} ((x-2)^2 - (\ln x)^2)$

$V = \pi \int_{1}^{3.14619} ((\ln x)^2 - (x-2)^2) + \pi \int_{.158594}^{2} ((x-2)^2 - (\ln x)^2)\,dx$

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1C1

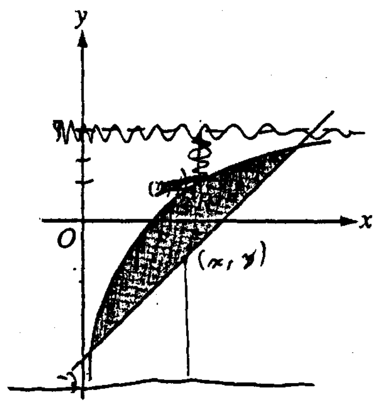
CALCULUS AB

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.



Work for problem 1(a)

$a = \cancel{\quad}$ $\int_0^{3.138} \ln x - (x-2)\,dx$

$\boxed{a \approx 1.80}$

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Continue problem 1 on page

1C2

Work for problem 1(b)

$v = ~~\#~~3 - y$

$v = 2\pi \int_0^{3.138} (3-(x-2))(\ln x - (x-2))\,dx$

$v = 2\pi \cdot 6.299$

$\boxed{v = 12.599\pi}$

Work for problem 1(c)

$v = 2\pi \int_0^{3.138} x(\ln x - (x-2))\,dx$

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GO ON TO THE NEXT PAGE.

Question 1

Overview

This problem gave two graphs that intersect at $x = 0.15859$ and $x = 3.14619$. A graphing calculator was required to find these two intersection values. Students needed to use integration to find an area and two volumes. In part (a) students had to find the area of the region bounded by the two graphs. In part (b) students had to calculate the volume of the solid generated by rotating the region about the horizontal line $y = -3$, a line that lies below the given region. Part (c) tested the students' ability to set up an integral for the volume of a solid generated by rotating the given region around a vertical axis, in this case the y-axis. The given functions could be solved for x in terms of y, leading to the use of horizontal cross sections in the shape of washers and an integral in terms of the variable y. Although no longer included in the *AP Calculus Course Description*, the method of cylindrical shells could also be used to write an integral expression for the volume in terms of the variable x.

Sample: 1A
Score: 9

The student earned all 9 points.

Sample: 1B
Score: 6

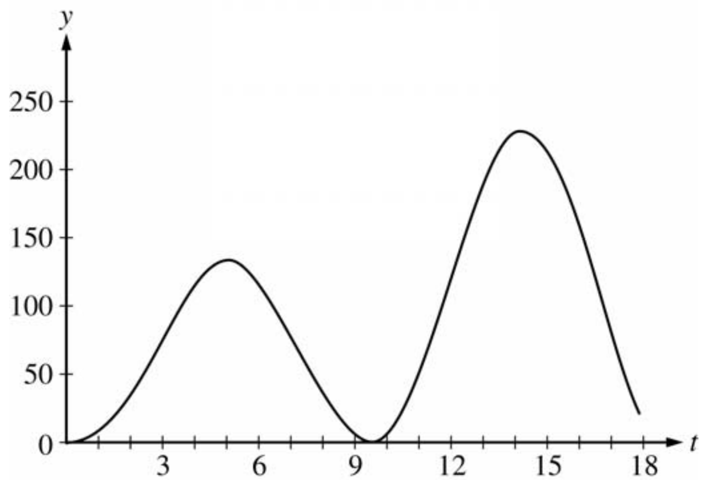
The student earned 6 points: 3 points in part (a) and 3 points in part (b). In part (a) the student has the correct integrand, which earned the first point. The definite integral has the correct limits to three decimal places, which earned the second point. The answer is correct to three decimal places and earned the third point. In part (b) the student has the correct integrand, which earned the first 2 points. The extra factor of -1 in each integrand term does not cause a problem since it is an equivalent form to the standard. The student correctly evaluates the integral and produces the correct answer to three decimal places. In part (c) the student does have a difference of squares, which provides entry into the problem, but since neither term is correct the student did not earn either integrand point. The student was not eligible for the limits/constant point since no integrand point was earned.

Sample: 1C
Score: 3

The student earned 3 points: 1 point in part (a) and 2 points in part (c). In part (a) the student has the correct integrand, which earned the first point. The limit point was not awarded because the student's limits are not correct to three decimal places. Since the lower limit of 0 is not within the acceptable range for limits, the student was not eligible for the answer point. In part (b) the student attempts a cylindrical shell setup, but the integrand is incorrect, so neither point was earned. Since neither integrand point was earned, the student was not eligible for the limits/constant/answer point. In part (c) the student correctly provides the integrand for the cylindrical shells method and earned the first 2 points. The student's limits, in particular the lower value of 0, are not in the acceptable range, so the limits/constant point was not earned.

Question 2

At an intersection in Thomasville, Oregon, cars turn left at the rate $L(t) = 60\sqrt{t}\sin^2\left(\frac{t}{3}\right)$ cars per hour over the time interval $0 \le t \le 18$ hours. The graph of $y = L(t)$ is shown above.



(a) To the nearest whole number, find the total number of cars turning left at the intersection over the time interval $0 \le t \le 18$ hours.

(b) Traffic engineers will consider turn restrictions when $L(t) \ge 150$ cars per hour. Find all values of t for which $L(t) \ge 150$ and compute the average value of L over this time interval. Indicate units of measure.

(c) Traffic engineers will install a signal if there is any two-hour time interval during which the product of the total number of cars turning left and the total number of oncoming cars traveling straight through the intersection is greater than 200,000. In every two-hour time interval, 500 oncoming cars travel straight through the intersection. Does this intersection require a traffic signal? Explain the reasoning that leads to your conclusion.

(a) $\int_0^{18} L(t)\,dt \approx 1658$ cars

$2: \begin{cases} 1: \text{setup} \\ 1: \text{answer} \end{cases}$

(b) $L(t) = 150$ when $t = 12.42831,\ 16.12166$
Let $R = 12.42831$ and $S = 16.12166$
$L(t) \ge 150$ for t in the interval $[R, S]$
$\frac{1}{S-R}\int_R^S L(t)\,dt = 199.426$ cars per hour

$3: \begin{cases} 1: t\text{-interval when } L(t) \ge 150 \\ 1: \text{average value integral} \\ 1: \text{answer with units} \end{cases}$

(c) For the product to exceed 200,000, the number of cars turning left in a two-hour interval must be greater than 400.

$\int_{13}^{15} L(t)\,dt = 431.931 > 400$

$4: \begin{cases} 1: \text{considers 400 cars} \\ 1: \text{valid interval } [h, h+2] \\ 1: \text{value of } \int_h^{h+2} L(t)\,dt \\ 1: \text{answer and explanation} \end{cases}$

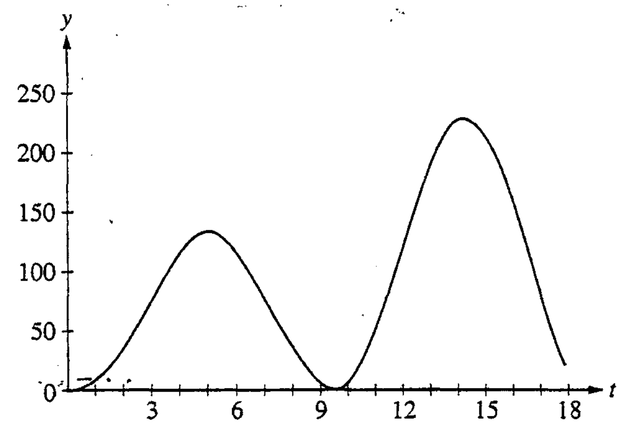
OR

The number of cars turning left will be greater than 400 on a two-hour interval if $L(t) \ge 200$ on that interval.
$L(t) \ge 200$ on any two-hour subinterval of $[13.25304, 15.32386]$.

OR

$4: \begin{cases} 1: \text{considers 200 cars per hour} \\ 1: \text{solves } L(t) \ge 200 \\ 1: \text{discusses 2 hour interval} \\ 1: \text{answer and explanation} \end{cases}$

Yes, a traffic signal is required.



Work for problem 2(a)

$L(t) = 60\sqrt{t}\sin^2\left(\frac{t}{3}\right)$

$\int_0^{18} 60\sqrt{t}\sin^2\left(\frac{t}{3}\right)dt$

$= 1657.8237$

$= 1658$ cars

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Continue problem 2 on page 7.

Work for problem 2(b)

$L(t) = 150 = 60\sqrt{t}\,\sin^2\left(\frac{t}{3}\right)$

$60\sqrt{t}\sin^2\frac{t}{3} - 150 = 0$

$t = 12.42831$ or 16.121657

$\frac{1}{b-a}\int_a^b L(t) = \frac{1}{16.121657-12.42831}\int_{12.42831}^{16.121657} 60\sqrt{t}\sin^2\frac{t}{3}\,dt$

$= \frac{1}{3.693347}\left[736.54986\right] = 199.426195$

$12.428 \le t \le 16.121657$ 199.426 cars/hr

Work for problem 2(c)

Cars turning left x oncoming cars going straight

(Cars turning left) 500 ≤ 200,000

cars turning left = 400

$\int_{14}^{16} L(t) = \int_{14}^{16} 60\sqrt{t}\,\sin^2\left(\frac{t}{3}\right)dt = 412.26$

Yes, there will need to be a signal because between the interval t=14 and t=16, 412 cars turn left. When you multiply that by 500, it exceeds 200,000

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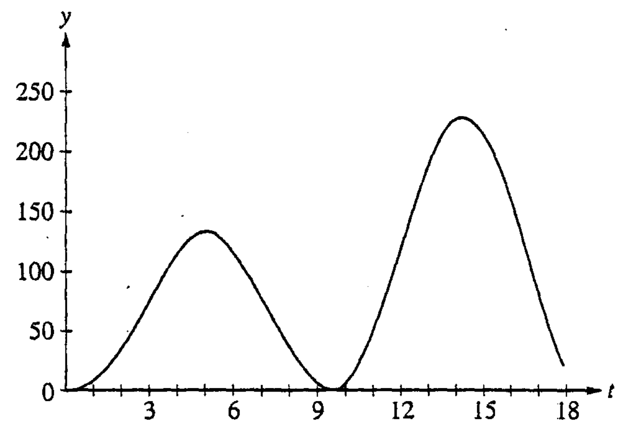

Work for problem 2(a)

$\int_0^{18} L(t)\,dt$

$\int_0^{18} \left[60\sqrt{t}\sin^2(t/3)\right]dt$

1658

1658 total cars turn left through the intersection between the hours of 0 hours and 18 hours.

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Continue problem 2 on page 7.

Work for problem 2(b)

$L(t) > 150$

$60\sqrt{t}\sin^2(t/3) > 150$

$t = [12.42831, 16.121657]$

$\boxed{\text{L(t) is greater than 150 at all hours between 12.42831 hours and 16.121657 hours.}}$

average value:

$\frac{1}{b-a}\int_a^b L(t)dt$

$\frac{1}{16.121657 - 12.42831}\int_{12.42831}^{16.121657} [60\sqrt{t}\sin^2(t/3)]\,dt$

199.426

$\boxed{\text{The average number of cars turning left between 12.42831 hours and 16.121657 hours is 199.426 cars.}}$

Work for problem 2(c)

500 straight cars/2 hours

200,000 = straight cars * left cars
(500)

$\frac{200{,}000}{500} = 400$

Therefore, 400 left turn cars during a 2-hour interval would require a trafficsignal.

$\int_a^{a+2} L(t)dt \geq 400?$

13.253, 15.323

$L(t) \geq 200?$

$L(t) \geq 200$ at

$[13.253, 15.323]$

$\boxed{\text{Yes, the intersection does require a traffic signal. At 13.253 hours, L(t) = 200 and increases upward. At at least 200 cars per hour, the product would exceed 200000 therefore requiring a signal. The flow of cars does not drop below 200 cars/hr until 15.323 hours.}}$

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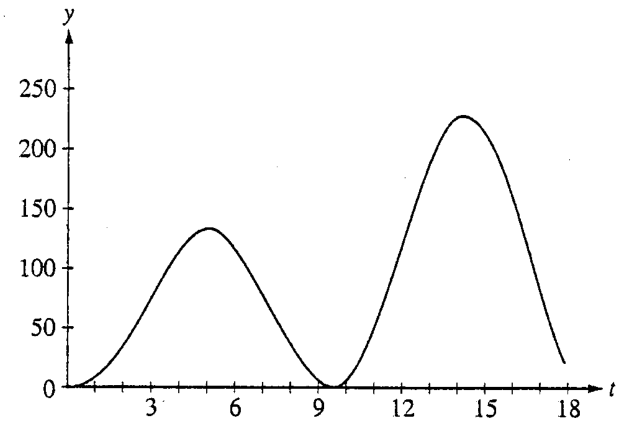

Work for problem 2(a)

total cars $\Rightarrow \int L(t)$

$= \int_0^{18} 60\sqrt{t}\, \sin^2\left(\frac{t}{3}\right)$

≈ 1658 cars

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Continue problem 2 on page 7.

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Work for problem 2(b)

$L(t) \geq 150$ from ≈ 12.428 to ≈ 16.120

$\frac{1}{b-a} \int_{12.428}^{16.120} 60\sqrt{t} \sin^2(\frac{t}{3})$

$\frac{1}{16.120-12.428} (736.34771)$

$\frac{1}{3.692} (736.34771) \approx 199.4$ = avg. # of cars turning left

Work for problem 2(c)

$\left(\text{total \# of cars turning left}\right)(500 \text{ straight cars}) \geq 200,000$?

$\left(\int_{12.428}^{16.120} 60\sqrt{t} \sin^2(\frac{t}{3})\right)(500)$

$(736.34771)(500) = 368173.855$

$368174 \geq 200,000$

$\boxed{\text{Yes}}$

The product of cars turning (L) and cars going straight is greater than 200,000, and requires the installation of a traffic signal.

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Question 2

Overview

This problem gave students a function that modeled the rate, in cars per hour, at which cars turn left at a given intersection. In part (a) students had to use the definite integral to find the total number of cars that turned left in a given time period. In part (b) students had to use their graphing calculators to find the time interval during which the rate equaled or exceeded 150 cars per hour and then compute the average value of the rate over this time interval. Part (c) described a condition that would require the installation of a traffic signal at an intersection and asked students to decide if a signal was necessary at this particular intersection. Students could do this in several ways. For example, students could recognize that a signal would be required if the number of cars that turn left over a two-hour time interval exceeds 400 cars and then find an appropriate interval. Or students might recognize that a signal would be required if the average value of the rate at which cars turn left over any two-hour time interval exceeds 200 cars per hour. Since the rate itself exceeds 200 cars per hour during the interval $13.253 < t < 15.324$, the average value of the rate will also exceed 200 cars per hour during this time interval of length greater than two hours, and thus a signal would be required.

Sample: 2A
Score: 9

The student earned all 9 points. In part (c) all 4 points were earned for a correct argument based on the value of the integral of $L(t)$ over a two-hour period.

Sample: 2B
Score: 6

The student earned 6 points: 2 points in part (a), 2 points in part (b), and 2 points in part (c). In part (a) the student provides the correct definite integral and its correct evaluation to the nearest whole number, which earned both points. In part (b) the student determines the correct interval, which earned the first point, and provides the correct average value setup to earn the second point. The third point was not earned because the units given for the computed average value are not correct. In part (c) the student earned the first point for observing that 400 left turns in a two-hour interval are needed. The second point was earned for correctly identifying the interval on which $L(t) \geq 200$. The student never compares the length of the given interval to 2 and thus did not earn the third point. The student was not eligible for the fourth point.

Sample: 2C
Score: 4

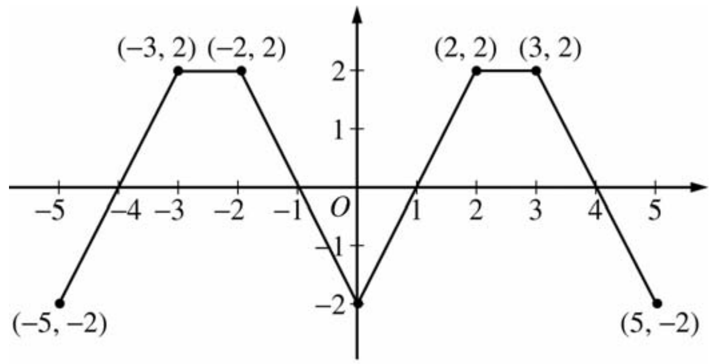
The student earned 4 points: 2 points in part (a), 1 point in part (b), and 1 point in part (c). In part (a) the student provides the correct definite integral and its correct evaluation to the nearest whole number, which earned both points. In part (b) the upper limit for the student's interval is not correct in the third decimal place, so the first point was not earned. The second point was earned for the average value setup since the student provides the correct value for the given integral, divided by the length of the interval. The third point was not earned because the student rounds the answer to the first decimal place instead of the required three. In part (c) the first point was earned for the observation that the product of the total number of cars turning left and 500 needs to be greater than 200,000. The student does not consider a two-hour interval, so no further points were earned.

Question 3

The graph of the function f shown above consists of six line segments. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$.

(a) Find $g(4)$, $g'(4)$, and $g''(4)$.

(b) Does g have a relative minimum, a relative maximum, or neither at $x = 1$? Justify your answer.



Graph of f

(c) Suppose that f is defined for all real numbers x and is periodic with a period of length 5. The graph above shows two periods of f. Given that $g(5) = 2$, find $g(10)$ and write an equation for the line tangent to the graph of g at $x = 108$.

(a) $g(4) = \int_0^4 f(t)\,dt = 3$

$g'(4) = f(4) = 0$

$g''(4) = f'(4) = -2$

$3 : \begin{cases} 1 : g(4) \\ 1 : g'(4) \\ 1 : g''(4) \end{cases}$

(b) g has a relative minimum at $x = 1$ because $g' = f$ changes from negative to positive at $x = 1$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

(c) $g(0) = 0$ and the function values of g increase by 2 for every increase of 5 in x.

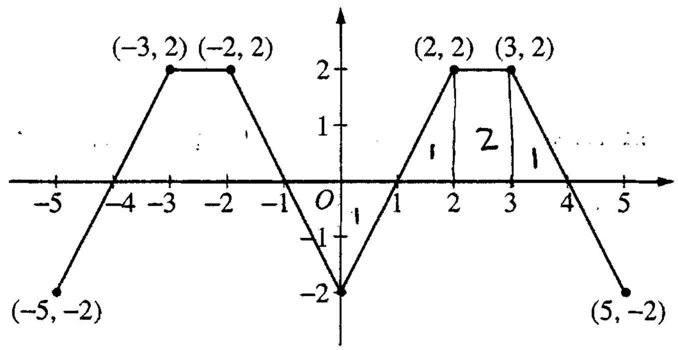
$g(10) = 2g(5) = 4$

$$g(108) = \int_0^{105} f(t)\,dt + \int_{105}^{108} f(t)\,dt$$
$$= 21g(5) + g(3) = 44$$

$g'(108) = f(108) = f(3) = 2$

An equation for the line tangent to the graph of g at $x = 108$ is $y - 44 = 2(x - 108)$.

$4 : \begin{cases} 1 : g(10) \\ 3 : \begin{cases} 1 : g(108) \\ 1 : g'(108) \\ 1 : \text{equation of tangent line} \end{cases} \end{cases}$



Graph of f

Work for problem 3(a)

$g(4) = \int_0^4 f(t)\,dt = 3$

$\boxed{g(4) = 3}$

$\boxed{g'(4) = f(4) = 0}$

$g''(4) = f'(4) = \frac{2+2}{3-5} = \frac{4}{-2} = -2$

$\boxed{g''(4) = f'(4) = -2}$

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Continue problem 3 on page 9.

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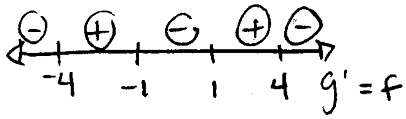
Work for problem 3(b)



g has a relative minimum at $x=1$ because $g'(x)=f(x)$ changes from negative to positive at $x=1$

Work for problem 3(c)

$g'(x) = f(x)$

$g(x) = \int f(x)\,dx$

- if $g(5)=2$ and f is periodic w/ a period length of 5, the $\boxed{g(10)=4}$
- $g(108)=?$

$g(108) = \int_0^{108} f(x)\,dx = \boxed{44}$

$g'(108) = f(108) = 2$

$\boxed{(y-44) = 2(x-108)}$

$y = 2x - 72$

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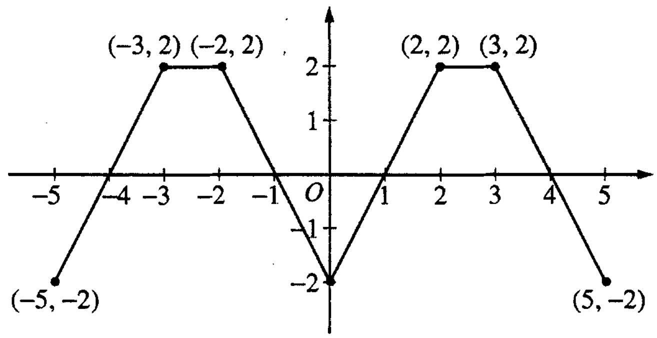
END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.



Graph of f

Work for problem 3(a)

$g(4) = \int_0^4 f(t)\,dt = 3$

$g'(4) = \frac{d}{dt}\int_0^{x} f(t)\,dt = f(x)$

$= f(4) = 0$

$g''(4) = f'(4) = \frac{-2-2}{5-3} = \frac{-4}{2} = -2$

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Continue problem 3 on page 9.

Work for problem 3(b)

$g(1) = \int_0^1 f(x)\,dt = -1$

$g(-5) = \int_0^{-5} f(x)\,dt = -2$

$g(-1) = \int_0^{-1} f(x)\,dt = +1$

$g(4) = \int_0^4 f(x)\,dt = 3$

$g(5) = \int_0^5 f(x)\,dt = 2$

g has neither a max or min @ $x = 1$ because max occurs @ $x = 4$ and min @ $x = -5$

Work for problem 3(c)

$\frac{10}{5} = 2$ $\quad g(5) = \int_0^5 f(t)\,dt = 2$

$g(10) = 2\int_0^5 f(t)\,dt$

$g(10) = 2(2) = 4$

$g(108) = \frac{108}{5}\int_0^5 f(t)\,dt = \frac{216}{5}$

$\frac{108}{5} = 21\ \frac{3}{5}$ $\quad f(108) = f(3)$

$g'(108) = f(108) = f(3) = 2$

$y - \frac{216}{5} = 2(x - 108)$

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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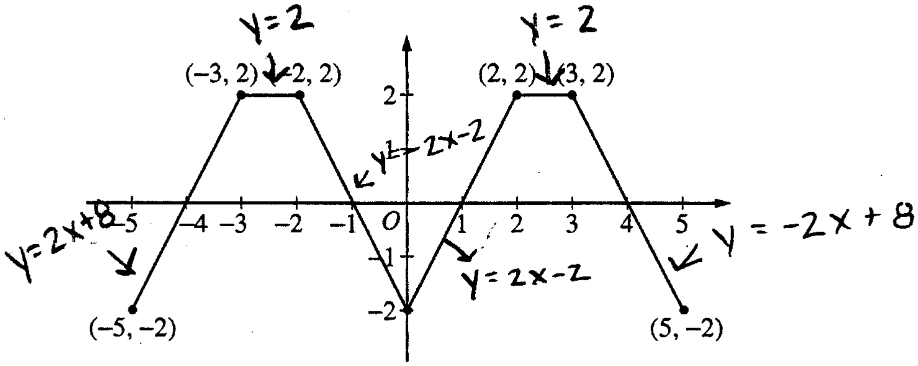

Graph of f

Work for problem 3(a)

$\frac{2-(-2)}{-3+5} = 2$ $y=2x+8$ $\frac{2-(-2)}{-2} = -2$ $y=-2x-2$

$\frac{2+2}{2} = 2$ $y=2x-2$ $\frac{2+2}{3-5} = -2$ $y=-2x+8$

$g(4) = \int_0^4 (-2x+8)\,dx$

$= -x^2+8x\Big]_0^4$

$= 16$

$\boxed{\therefore g(4) = 16}$

$\boxed{g'(4) = 0}$

$\boxed{g''(4) = -2}$

Continue problem 3 on page 9.

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Work for problem 3(b)

$$\begin{array}{cccccccc} - & \circ & + & \circ & - & \circ & + & \circ \\ & -4 & & -1 & & 1 & & 4 \end{array}$$

There is a relative minimum at $x=1$ because the sign analysis shows that slope is negative from $(-1, 1)$ and then there is 0 slope at $x=1$ and slope is positive from $(1, 4)$.

Work for problem 3(c)

slope $= -2$

$2 = -2(5) + b$

$b = 12$

$\therefore y = -2x + 12$

The equation of the line tangent to the curve is $y = -2x + 12$

$y|_{x=10} = -8$

Do not write beyond this border.

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

Question 3

Overview

This problem required the Fundamental Theorem of Calculus. Students were given the piecewise-linear graph of the function f and were asked about the function g defined as the definite integral of f from 0 to x. It was expected that students would use the graph of f, as well as the area bounded by the graph of f and the x-axis, to answer questions about g, g', and g''. Part (a) asked for the values of $g(4)$, $g'(4)$, and $g''(4)$. Part (b) asked about the behavior of g at $x = 1$. In part (c) the function f is extended in a periodic fashion. Students had to compute $g(10)$, $g(108)$, and $g'(108)$ using the periodic behavior of f.

Sample: 3A
Score: 9

The student earned all 9 points. In part (c) the student writes an equation $y = 2x - 72$ but declares the correct answer for the equation of the tangent line by enclosing it in a box.

Sample: 3B
Score: 6

The student earned 6 points: 3 points in part (a) and 3 points in part (c). In part (a) the student gives correct values for $g(4)$, $g'(4)$, and $g''(4)$, which earned 3 points. In part (b) the student claims that there is neither a maximum nor a minimum at $x = 1$; consequently, the student was not eligible for the justification point. In part (c) the student earned 1 point for giving the correct value for $g(10)$. The student declares an incorrect value for $g(108)$ but a correct value for $g'(108)$. These values are correctly used to write the equation of the tangent line. The student earned 1 point for $g'(108)$ and 1 point for the tangent line since the incorrect value for $g(108)$ is used correctly.

Sample: 3C
Score: 3

The student earned 3 points: 2 points in part (a) and 1 point in part (b). In part (a) the student correctly identifies $g'(4)$ and $g''(4)$, which earned 2 points. In part (b) the student declares a relative minimum at $x = 1$ but did not earn the justification point. The argument refers to slope without designating whether it refers to the slope of f or the slope of g. No points were earned in part (c).

Question 4

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \le t \le 80$ seconds, as shown in the table above.

(a) Find the average acceleration of rocket A over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\,dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\,dt$.

(c) Rocket B is launched upward with an acceleration of $a(t) = \dfrac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.

(a) Average acceleration of rocket A is $\dfrac{v(80)-v(0)}{80-0} = \dfrac{49-5}{80} = \dfrac{11}{20}\ \text{ft}/\text{sec}^2$	1 : answer
(b) Since the velocity is positive, $\int_{10}^{70} v(t)\,dt$ represents the distance, in feet, traveled by rocket A from $t = 10$ seconds to $t = 70$ seconds. A midpoint Riemann sum is $20[v(20)+v(40)+v(60)]$ $= 20[22+35+44] = 2020$ ft	$3 : \begin{cases} 1 : \text{explanation} \\ 1 : \text{uses } v(20),\ v(40),\ v(60) \\ 1 : \text{value} \end{cases}$
(c) Let $v_B(t)$ be the velocity of rocket B at time t. $v_B(t) = \int \dfrac{3}{\sqrt{t+1}}\,dt = 6\sqrt{t+1} + C$ $2 = v_B(0) = 6 + C$ $v_B(t) = 6\sqrt{t+1} - 4$ $v_B(80) = 50 > 49 = v(80)$ Rocket B is traveling faster at time $t = 80$ seconds.	$4 : \begin{cases} 1 : 6\sqrt{t+1} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{finds } v_B(80)\text{, compares to } v(80)\text{,} \\ \quad\ \text{and draws a conclusion} \end{cases}$
Units of ft/sec^2 in (a) and ft in (b)	1 : units in (a) and (b)

NO CALCULATOR ALLOWED

CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

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Work for problem 4(a)

$\text{avgval}_{a(t)} = \dfrac{v(b)-v(a)}{b-a}$

$\text{avgval}_{a(t)} = \dfrac{v(80)-v(0)}{80-0}$

$\text{avgval}_{a(t)} = \dfrac{49-5}{80-0}$

$\boxed{\text{avgval}_{a(t)} = 11/20 \text{ ft/s}^2}$

Work for problem 4(b)

$\int_{10}^{70} v(t)\,dt$ gives the distance the rocket has traveled from time 10 seconds to 70 seconds in feet.

$\Delta x = \dfrac{70-10}{3} = 20$

$p(t) = 20\left[v(20)+v(40)+v(60)\right]$

$p(t) = 20[22+35+44] = 20[101]$

$\boxed{p(t) = 2020 \text{ feet}}$

Continue problem 4 on page 11.

NO CALCULATOR ALLOWED

Work for problem 4(c)

Rocket B

$a(t) = \frac{3}{\sqrt{t+1}}$

$v(t) = \int \frac{3}{\sqrt{t+1}}\, dt$

let $u = t+1$

$\frac{du}{dx} = 1$

$du = dx$

$v(t) = 3\int u^{-1/2}\, du$

$v(t) = \frac{3u^{1/2}}{1/2} + c$

$v(t) = 6(t+1)^{1/2} + c$

$2 = 6(0+1)^{1/2} + c$

$c = -4$

$v(t) = 6(t+1)^{1/2} - 4$

$v(80) = 6(80+1)^{1/2} - 4$

$v(80) = 6\sqrt{81} - 4$

$v(80) = 54 - 4$

$\boxed{v(80) = 50 \text{ ft/sec}}$

Rocket A

$\boxed{v(80) = 49 \text{ ft/sec}}$

Rocket B is traveling faster at t = 80 sec. Rocket B's velocity was found by $v(t) = \int a(t)\,dt$ and is 50 ft/sec. Rocket A's velocity was 49 ft/sec

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NO CALCULATOR ALLOWED

CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

Work for problem 4(a)

$$\frac{1}{80}\left(9+8+7+6+5+4+3+2\right)=$$

$$=\frac{46}{80}=\frac{23}{40}\text{ ft/sec}^2$$

Work for problem 4(b)

$\int_{10}^{70} v(t)\,dt$ indicates the distance the rocket traveled over the interval $10 \le t \le 70$. In this case, it indicates total distance because the rocket's velocity is only increasing during this time period.

$$(22+2(35)+44)=136\text{ ft.}$$

$$\begin{array}{r} 70 \\ 22 \\ -44 \\ \hline 136 \end{array}$$

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Continue problem 4 on page 11.

Work for problem 4(c)

$3(t+1)^{-1/2}$

$v(t) = 6\sqrt{t+1} + C$

$2 = 6\sqrt{0+1} + c$

$2 = 6 + c$

$-4 = c$

$v(t) = 6\sqrt{t+1} - 4$

$v(80) = 6\sqrt{80+1} - 4$

$6 \cdot 9 - 4$

$v(80) = 52$ ft/sec

Rocket B is traveling faster at t=80. The table states that Rocket A is traveling at 49 ft/sec. By taking the antideravitive for a(t) of Rocket B and solving with initial conditions for C, then substituting t=80, we find v(80)=52 ft/sec, faster than Rocket A.

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 4C, 1

NO CALCULATOR ALLOWED

CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

Work for problem 4(a)

$\text{acceleration}_{\text{ave}} = \frac{11}{20}$ feet/second^2

$$\begin{array}{r} 49 \\ -5 \\ \hline 44 \end{array} \qquad \frac{44}{80} \qquad \frac{11}{20}$$

Work for problem 4(b)

The integral of the velocity is the position. $\int_{10}^{70} v(t)\,dt$ means the position of the rocket from 10 sec to 70 sec.

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Continue problem 4 on page 11.

NO CALCULATOR ALLOWED

Work for problem 4(c)

$a(t) = \frac{3}{\sqrt{t+1}}$

$t+1 = u$

$1\,dt = du$

$\int \frac{3}{\sqrt{t+1}}\,dt$

$= \int \frac{3\,du}{\sqrt{u}}$

$= \int 3u^{-\frac{1}{2}}\,du$

$= 3 \cdot 2u^{\frac{1}{2}}$

$= 6u^{\frac{1}{2}}$

$v(t) = 6\sqrt{t+1}$

$v(80) = 6\sqrt{80+1}$

$= 6\sqrt{81}$

$= 6 \times 9$

$= 54$ feet/second.

Rocket B is traveling faster at time = 80 sec.

The antiderivative of the acceleration gives the velocity. Using this, the velocity of Rocket B was discovered to be 54 feet per second at time = 80 seconds. Compared to the velocity of Rocket A, which is 49 feet per second, Rocket B is traveling faster.

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Question 4

Overview

This problem presented students with a table of velocity values for rocket A at selected times. In part (a) students needed to recognize the connection between the average acceleration of the rocket over the given time interval and the average rate of change of the velocity over this interval. In part (b) students had to recognize the definite integral as the total change, in feet, in rocket A's position from time $t = 10$ seconds to time $t = 70$ seconds and then approximate the value of this definite integral using a midpoint Riemann sum and the data in the table. Units of measure were important in both parts (a) and (b). Part (c) introduced a second rocket and gave its acceleration in symbolic form. The students were asked to compare the velocities of the two rockets at time $t = 80$ seconds. The velocity of rocket B could be determined by finding the antiderivative of the acceleration and using the initial condition or by using the Fundamental Theorem of Calculus and computing a definite integral.

Sample: 4A
Score: 9

The student earned all 9 points.

Sample: 4B
Score: 6

The student earned 6 points: 2 points in part (b), 3 points in part (c), and the units point. The first line in part (a) is a correct but uncommon method. An error was made in the addition. In part (b) the explanation is acceptable; the student correctly identifies the midpoint values but the method is incorrect. In part (c) the fourth point was not earned due to an error in the computation of $v(80)$ for rocket B.

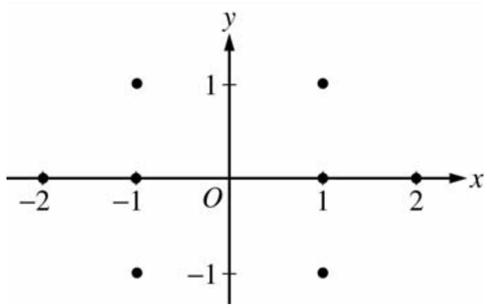
Sample: 4C
Score: 3

The student earned 3 points: 1 point in part (a) and 2 points in part (c). The units are correct in part (a) but missing in part (b). For the explanation in part (b), the integral does not represent the position but rather the displacement over the time interval $[10, 70]$. The constant of integration does not appear in part (c) so the student earned the antiderivative and comparison points only.

Question 5

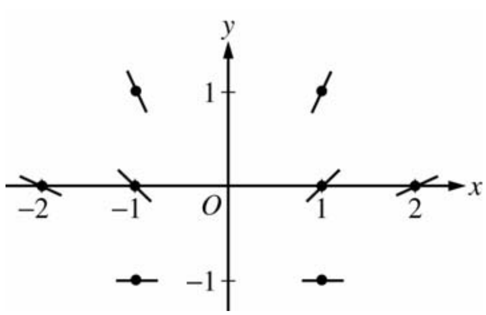
Consider the differential equation $\dfrac{dy}{dx} = \dfrac{1+y}{x}$, where $x \neq 0$.

(a) On the axes provided, sketch a slope field for the given differential equation at the eight points indicated. **(Note: Use the axes provided in the pink exam booklet.)**



(b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(-1) = 1$ and state its domain.

(a)



2 : sign of slope at each point and relative steepness of slope lines in rows and columns

(b) $\dfrac{1}{1+y}\,dy = \dfrac{1}{x}\,dx$

$\ln|1+y| = \ln|x| + K$

$|1+y| = e^{\ln|x| + K}$

$1 + y = C|x|$

$2 = C$

$1 + y = 2|x|$

$y = 2|x| - 1$ and $x < 0$

or

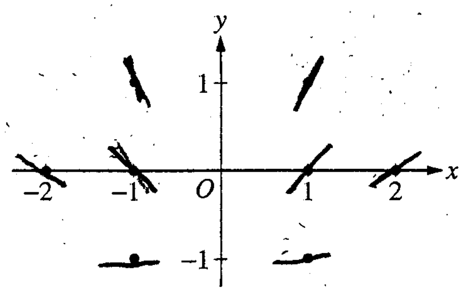
$y = -2x - 1$ and $x < 0$

7 :
- 6 :
  - 1 : separates variables
  - 2 : antiderivatives
  - 1 : constant of integration
  - 1 : uses initial condition
  - 1 : solves for y
- Note: max $3/6$ [1-2-0-0-0] if no constant of integration
- Note: $0/6$ if no separation of variables
- 1 : domain

5A1

NO CALCULATOR ALLOWED

Work for problem 5(a)



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Continue problem 5 on page 13.

NO CALCULATOR ALLOWED

Work for problem 5(b)

$$\frac{dy}{dx} = \frac{1+y}{x}$$

$$\int \frac{dy}{1+y} = \int \frac{dx}{x}$$

$$e^{\ln|1+y|} = e^{(\ln|x|+C)}$$

$1+y = Cx \qquad 1+y > 0$

$y = Cx - 1, \qquad y > -1$

$f(x) = Cx - 1, \qquad f(x) > -1$

$f(-1) = C(-1) - 1 = 1$

$C(-1) = 2$

$C = -2$

$\boxed{f(x) = -2x - 1}, \quad f(x) > -1$

$D = \{x \in \mathbb{R} \mid x < 0\} \quad -2x - 1 > -1$

or

$D = (-\infty, 0) \qquad -2x > 0$

$2x < 0 \quad x < 0$

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NO CALCULATOR ALLOWED

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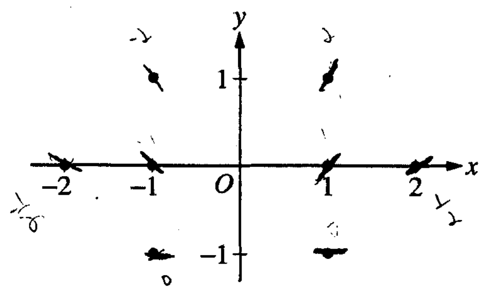
Work for problem 5(a)



$y' = \frac{1+y}{x}$

$(1,0)$ $\frac{1}{1} = 1$ $\underline{m=1}$

$(1,1)$ $\frac{2}{1} = 2$ $\underline{m=2}$

$(1,-1)$ $\frac{0}{1} = 0$ $\underline{m=0}$

$(2,0)$ $\frac{1}{2} = \frac{1}{2}$ $\underline{m=\frac{1}{2}}$

$(-1,0)$ $\frac{1}{-1} = -1$ $\underline{m=-1}$

$(-1,1)$ $\frac{2}{-1} = -2$ $\underline{m=-2}$

$(-1,-1)$ $\frac{0}{-1} = 0$ $\underline{m=0}$

$(-2,0)$ $\frac{1}{-2} = -\frac{1}{2}$ $\underline{m=-\frac{1}{2}}$

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Continue problem 5 on page 13.

 5B2

NO CALCULATOR ALLOWED

Work for problem 5(b)

$y' = \frac{1+y}{x}$

$\int \frac{dy}{1+y} = \int \frac{dx}{x}$

$\ln_e(1+y) = \ln x + C$

$e^{(\ln x + C)} = 1+y$

$y = Ce^{\ln x} - 1$ where $x \neq 0$

$Ce^{\ln x} \neq 1$ $x = 0$

$1 = Ce^{\ln(-1)} - 1$ $2 = Ce^{\ln -1}$

$C = \frac{2}{e^{\ln(-1)}}$

$y = \frac{2}{e^{\ln(-1)}}(e^{\ln x}) - 1$

domain: $(-\infty, 0)(0, \infty)$

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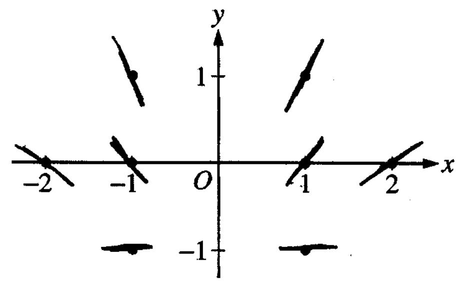
5C1

NO CALCULATOR ALLOWED

Work for problem 5(a)



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Continue problem 5 on page 13.

Work for problem 5(b)

$u = 1+y$
$du = 1\,dy$

$\int \frac{1}{1+y}\,dy = \int x\,dx$

$\int \frac{1}{u}\,du = \int x\,dx$

$\ln u = \frac{1}{2}x^2$

$\ln 1+y = \frac{1}{2}x^2$

$\ln 1+1 = \frac{1}{2}$

$\ln 2 = \frac{1}{2}$

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Question 5

Overview

This problem presented students with a separable differential equation. In part (a) students were asked to sketch its slope field at eight points. Part (b) required solving the separable differential equation to find the particular solution with $f(-1) = 1.$ Students were also asked for the domain of the solution to this differential equation. This is an important consideration when solving any differential equation and in particular when the differential equation is not defined for all values of the independent and/or dependent variables. Students needed to recognize for this equation that the particular solution must be a differentiable function on an open interval that contains $x = -1$ and does not contain $x = 0.$

Sample: 5A
Score: 9

The student earned all 9 points. The statement $y + 1 > 0$ is used by the student to explain why $\ln|1 + y|$ can be replaced by $\ln(1 + y).$

Sample: 5B
Score: 6

The student earned 6 points: 2 points in part (a) and 4 points in part (b). The student's slope field in part (a) is correct and earned 2 points. The computations were ignored. In part (b) the separation point was earned with the student's second line. The next line earned 1 antidifferentiation point for $\ln(1 + y)$ and no points for $\ln(x).$ The constant of integration point was also earned. The initial condition point was earned in the work on the right side of the page. The $\ln(-1)$ prevented the student from being eligible for the solution point. The domain point was not earned.

Sample: 5C
Score: 3

The student earned 3 points: 2 points in part (a) and 1 point in part (b). The student's slope field in part (a) is correct and earned 2 points. In part (b) the incorrect separation of variables did not earn a point. Since the separation is incorrect, the student was eligible for only 1 antidifferentiation point if both antidifferentiations were correct. Here that occurs. However, the y antidifferentiation is not finished until $\ln(1 + y)$ first appears, and this earned the point. The absence of a constant of integration prevented the student from earning further points. The incorrect separation also implies that the student was not eligible for the solution point. If the constant of integration point had been earned, then the initial condition point would not have been earned since $1 + y \neq -1$ if $y = 1.$

Question 6

The twice-differentiable function f is defined for all real numbers and satisfies the following conditions:

$$f(0) = 2, \quad f'(0) = -4, \text{ and } f''(0) = 3.$$

(a) The function g is given by $g(x) = e^{ax} + f(x)$ for all real numbers, where a is a constant. Find $g'(0)$ and $g''(0)$ in terms of a. Show the work that leads to your answers.

(b) The function h is given by $h(x) = \cos(kx) f(x)$ for all real numbers, where k is a constant. Find $h'(x)$ and write an equation for the line tangent to the graph of h at $x = 0$.

(a) $g'(x) = ae^{ax} + f'(x)$
$g'(0) = a - 4$

$g''(x) = a^2 e^{ax} + f''(x)$
$g''(0) = a^2 + 3$

$$4 : \begin{cases} 1 : g'(x) \\ 1 : g'(0) \\ 1 : g''(x) \\ 1 : g''(0) \end{cases}$$

(b) $h'(x) = f'(x)\cos(kx) - k\sin(kx) f(x)$
$h'(0) = f'(0)\cos(0) - k\sin(0) f(0) = f'(0) = -4$
$h(0) = \cos(0) f(0) = 2$
The equation of the tangent line is $y = -4x + 2$.

$$5 : \begin{cases} 2 : h'(x) \\ 3 : \begin{cases} 1 : h'(0) \\ 1 : h(0) \\ 1 : \text{equation of tangent line} \end{cases} \end{cases}$$

NO CALCULATOR ALLOWED

Work for problem 6(a)

$g(x) = e^{ax} + f(x)$

$g'(x) = a.e^{ax} + f'(x)$. $\Rightarrow g'(0) = a.e^{0} + f'(0)$

$= a - 4$

$g''(x) = a^2 e^{ax} + f''(x) \Rightarrow g''(0) = a^2.e^{0} + f''(0)$

$= a^2 + 3$

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Continue problem 6 on page 15.

NO CALCULATOR ALLOWED

Work for problem 6(b)

$h(x) = \cos(kx) f(x)$

$h'(x) = -\sin(kx) . k . f(x) + \cos(kx) . f'(x)$

$= -k . \sin(kx) . f(x) + \cos(kx) . f'(x)$

$h'(0) = -k . \sin 0 . f(0) + \cos(0) . f'(0)$

$= 0 + 1 . (-4) = -4$

$y = mx + b \qquad h(0) = \cos 0 . f(0) = 1 . \alpha = \alpha$

$y = -4x + b$

$\alpha = -4.0 + b$

$b = \alpha$

the line tangent to the graph of h is $y = -4x + \alpha$

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STOP

END OF EXAM

THE FOLLOWING INSTRUCTIONS APPLY TO THE COVERS OF THE SECTION II BOOKLET.

- **MAKE SURE YOU HAVE COMPLETED THE IDENTIFICATION INFORMATION AS REQUESTED ON THE FRONT AND BACK COVERS OF THE SECTION II BOOKLET.**
- **CHECK TO SEE THAT YOUR AP NUMBER LABEL APPEARS IN THE BOX(ES) ON THE COVER(S).**
- **MAKE SURE YOU HAVE USED THE SAME SET OF AP NUMBER LABELS ON ALL AP EXAMS YOU HAVE TAKEN THIS YEAR.**

NO CALCULATOR ALLOWED

Work for problem 6(a)

$u = ax$
$du = a$

$g(x) = e^{ax} + f(x)$

$g'(x) = ae^{ax} + f'(x)$

$g'(0) = ae^{a0} - 4$

$g'(0) = a - 4$

$g''(0) = 2ae^{ax} + f''(0)$

$g''(0) = 2ae^{a(0)} + 3$

$g''(0) = 2a + 3$

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Continue problem 6 on page 15.

Work for problem 6(b)

$h(x) = \cos(kx) f(x)$

$h'(x) = \cos(kx) f'(x) + k\sin(kx) f(x)$

$h'(0) = \cos(0) \cdot -4 + k\sin(0)\, 2$

$h'(0) = -4 + 0$

$h'(0) = -4$

$h(0) = \cos(k0) f(0)$

$h(0) = 1 \cdot 2$

$h(0) = 2$

$y - 2 = -4(x - 0)$

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STOP

END OF EXAM

THE FOLLOWING INSTRUCTIONS APPLY TO THE COVERS OF THE SECTION II BOOKLET.

- **MAKE SURE YOU HAVE COMPLETED THE IDENTIFICATION INFORMATION AS REQUESTED ON THE FRONT AND BACK COVERS OF THE SECTION II BOOKLET.**
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- **MAKE SURE YOU HAVE USED THE SAME SET OF AP NUMBER LABELS ON ALL AP EXAMS YOU HAVE TAKEN THIS YEAR.**

NO CALCULATOR ALLOWED

Do not write beyond this border.

Work for problem 6(a)

$f(0)=2 \quad f'(0)=-4 \quad f''(0)=3$

$g(x)=e^{ax}+f(x)$

$e^x \frac{d}{dx}=e^x$

$e^{f(x)}\frac{d}{dx}=e^{f(x)} f(x)\frac{d}{dx}$

$g'(x)=e^{ax}\cdot a+f'(x)$

$g'(x)=ae^{ax}+-4$

$g'(x)=ae^{ax}-4$

$e^0=1$

$g'(0)=a\,e^{a(0)}{}^{=1}-4$

$\boxed{g'(0)=a-4}$

$g''(x)=[ae^{ax}\cdot a]$

$g''(x)=a^2e^{ax}$

$g''(0)=a^2e^{a(0)}{}^{=1}$

$\boxed{g''(0)=a^2}$

Continue problem 6 on page 15.

NO CALCULATOR ALLOWED

Work for problem 6(b)

$h(x) = \cos(kx)\, f(x)$

$h'(x) = -\sin(kx) \cdot k \cdot f'(x)$

$\boxed{h'(x) = -k\, f(x) \sin(kx)}$ $(0,$

$h(x) = \cos(k(0))\, f(0)$

$= \cos 0 \cdot 2$

$= 2$

point $(0,2)$

$y - 2 = (-k\, f(x) \sin(kx))(x - 0)$

$y = [-k\, f(x) \sin(kx)](x) + 2$

Do not write beyond this border.

Do not write beyond this border.

STOP

END OF EXAM

THE FOLLOWING INSTRUCTIONS APPLY TO THE COVERS OF THE SECTION II BOOKLET.

- **MAKE SURE YOU HAVE COMPLETED THE IDENTIFICATION INFORMATION AS REQUESTED ON THE FRONT AND BACK COVERS OF THE SECTION II BOOKLET.**
- **CHECK TO SEE THAT YOUR AP NUMBER LABEL APPEARS IN THE BOX(ES) ON THE COVER(S).**
- **MAKE SURE YOU HAVE USED THE SAME SET OF AP NUMBER LABELS ON ALL AP EXAMS YOU HAVE TAKEN THIS YEAR.**

Question 6

Overview

This problem gave students the values of $f(0)$, $f'(0)$, and $f''(0)$ for a twice-differentiable function f. In part (a) the function g was defined as the sum of f and an exponential function involving a parameter. Students had to use the chain rule and addition rule for differentiation, and the given information about f, to compute $g'(0)$ and $g''(0)$ in terms of that parameter. Part (b) introduced a function h as the product of f and a cosine function involving the parameter k. Here students had to use the chain rule and product rule to compute the derivative of h and then use that derivative to write an equation for the line tangent to the graph of h at $x = 0$. Although not asked, it was hoped that the students would make the interesting observation that the equation of the tangent line at $x = 0$ is the same for all values of the parameter k.

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 2 points in part (a) and 4 points in part (b). In part (a) the student correctly presents $g'(x)$ and $g'(0)$. The student presents an incorrect $g''(x)$ and was not eligible for the fourth point in part (a). In part (b) the student's $h'(x)$ includes a sign error and earned only 1 of the 2 derivative points. The presented value for $h(0)$ is correct, $h'(0)$ is consistent with the student's $h'(x)$, and the student correctly writes an equation of the tangent line.

Sample: 6C
Score: 3

The student earned 3 points: 2 points in part (a) and 1 point in part (b). In part (a) the student correctly presents $g'(x)$ and $g'(0)$. The student presents an incorrect $g''(x)$ and was not eligible for the fourth point in part (a). In part (b) the student presents an incorrect $h'(x)$. The $h(0)$ point was earned. The student does not find $h'(0)$ and does not write an equation of the tangent line.